THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Mar 14

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This note only covers Sections 4.1-4.2 and it focuses on the definition of limit of a function and some basic limit theorems. There are many more results and problems you should be familiar with. If necessary, I will select some of them and discuss more specific examples next week, when you have learned one-sided limits, infinite limits, limits at infinity and two important limits.

Part I: Problems

1. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Suppose c is a cluster point of A.

   (a) Prove the **Sequential Criterion** for limit: $\lim_{x \to c} f(x) = L \in \mathbb{R}$ if and only if for every sequence in $A \setminus \{c\}$ converging to c, the sequence $(f(x_n))$ converges to L.

   (b) Suppose $\lim_{x \to c} f(x)$ does not exist. Show that there exists $\varepsilon_0 > 0$ and two sequences $(x_n), (y_n)$ in $A \setminus \{c\}$, both converging to c, such that $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

   (c) Prove the **Cauchy Criterion** for limit: $\lim_{x \to c} f(x)$ exists if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in A$ with $0 < |x - c|, |y - c| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Proof: (a) Please refer to **Theorem 4.1.8** in the text.

(b) From the **Divergence Criteria**, $f(x)$ does not have a limit at c if and only if there exists a sequence in $A \setminus \{c\}$ such that $\lim z_n = c$ while the sequence $(f(z_n))$ does not converge. Then by **Cauchy Criterion for sequence**, $\exists\, \varepsilon_0 > 0$ and two subsequences $(x_n), (y_n)$ of (z_n) such that $\lim x_n = \lim y_n = c$ and

$$|f(x_n) - f(y_n)| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$

(c) "$\Longrightarrow$": easy to prove.

"$\Longleftarrow$": this is an immediate consequence of (b).

If otherwise $\lim_{x \to c} f(x)$ does not exist, then by (b) we have that there exists $\varepsilon_0 > 0$ and two sequences $(x_n), (y_n)$ in $A \setminus \{c\}$, both converging to c, such that $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Now for this ε_0, there exists $\delta > 0$ such that whenever $x, y \in A$ with $0 < |x - c|, |y - c| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

However, since $\lim x_n = \lim y_n = c$, there exists $N \in \mathbb{N}$ such that

$$0 < |x_n - c| < \delta, \quad 0 < |y_n - c| < \delta$$

whenever $n \geq N$ ($x_n, y_n \neq c$ from our construction). Then from above we know $|f(x_n) - f(y_n)| < \varepsilon_0$, which is a contradiction.

2. **(Ex 4.1.15)** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } \ x \in \mathbb{Q} \\ 0 & \text{if } \ x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

   (a) Show that f has a limit at $x = 0$.

   (b) Show that f does not have a limit at any $c \neq 0$.

   Proof: (a) Notice that

$$|f(x)| = \begin{cases} |x| & \text{if } \ x \in \mathbb{Q} \\ 0 & \text{if } \ x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

   which implies $|f(x) - 0| \leq |x|, \forall x \in \mathbb{R}$.

   Therefore, $\forall \varepsilon > 0$, we choose $\delta(\varepsilon) = \varepsilon$ and then whenever $0 < |x - 0| < \delta = \varepsilon$, it follows that

$$|f(x) - 0| \leq |x| < \varepsilon.$$

   This is right the definition for $\lim_{x \to 0} f(x) = 0$.

   (b) When $c \neq 0$, (recall what we have learned in Chapter 2) we can find two sequences $(x_n) \subset \mathbb{Q}, (y_n) \subset \mathbb{R} \setminus \mathbb{Q}$ such that $\lim x_n = \lim y_n = c$. Then

$$\lim f(x_n) = \lim x_n = c \quad \text{while} \quad \lim y_n = \lim 0 = 0.$$

   From the Divergence Criteria, $\lim_{x \to c} f(x)$ does not exist.

   Remark: Compare this example with the **Dirichlet function**:

$$f(x) = \begin{cases} 1 & \text{if } \ x \in \mathbb{Q} \\ 0 & \text{if } \ x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

3. The definition of **Riemann function (or Thomae's function)** $R(x)$ is

$$R(x) = \begin{cases} \dfrac{1}{q} & \text{if } \ x = \dfrac{p}{q} \in \mathbb{Q} \ \text{ where } \ p, q \in \mathbb{Z}, q > 0 \ \text{ and } \ \gcd(p, q) = 1, \\ 0 & \text{if } \ x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

   Show that $\lim_{x \to c} R(x) = 0, \quad \forall c \in \mathbb{R}$. Notice that $R(0) = 1$ since we can write $0 = \dfrac{0}{1}$.

   Proof: WLOG, we only consider $c \geq 0$ and it suffices to prove that $\forall \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that whenever $0 < |x - c| < \delta$, we have

$$|R(x) - 0| = R(x) < \varepsilon.$$

   First we set $\delta_1 = 1$ and consider all the real numbers x with $0 < |x - c| < \delta_1 = 1$.

   If c is irrational, then $R(c) = 0 < \varepsilon$.

   If $c = \dfrac{p}{q}$ is rational, then $R(c) = \dfrac{1}{q}$. Notice that $R(c) \geq \varepsilon \Rightarrow q \leq \dfrac{1}{\varepsilon} \Rightarrow q \in S := \left\{1, 2, \cdots, \left[\dfrac{1}{\varepsilon}\right]\right\}$. So we pick out all the rational numbers $\dfrac{p}{q}$ in $(c-1, c+1) \setminus \{c\}$ with $q \in S$

and denote them by $x_1, x_2, \cdots, x_k$ **(think about why the number of such rational numbers is finite)**. If we put

$$\delta = \min\{\delta_1, |x_1 - c|, |x_2 - c|, \cdots, |x_k - c|\},$$

then whenever $0 < |x - c| < \delta$ we have $R(x) < \varepsilon$.

Therefore, $\lim_{x \to c} R(x) = 0$.

4. **(Composite function)**. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ and $x_0, y_0, L \in \mathbb{R}$. If

   (a) $\lim_{x \to x_0} g(x) = y_0$ and $\lim_{y \to y_0} f(y) = L$;

   (b) there exists $\delta > 0$ such that $g(x) \neq y_0$ whenever $0 < |x - x_0| < \delta$,

   show that $\lim_{x \to x_0} f(g(x)) = L$. Can we drop the condition (b)?

   Solution: From (a), $\forall \varepsilon > 0$, there exists $\gamma(\varepsilon) > 0$ such that whenever $0 < |y - y_0| < \gamma$ we have

$$|f(y) - L| < \varepsilon.$$

   For this $\gamma(\varepsilon)$, there exists $\delta_1 > 0$ such that whenever $0 < |x - x_0| < \delta_1$ we have

$$|g(x) - y_0| < \gamma.$$

   Now we rewrite (b) as that δ replaced by δ_2 and take $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $0 < |x - x_0| < \delta (\leq \delta_1, \delta_2)$ we have

$$g(x) \neq y_0 \quad \text{and} \quad |g(x) - y_0| < \gamma \Longrightarrow |f(g(x)) - L| < \varepsilon.$$

   Therefore, $\lim_{x \to x_0} f(g(x)) = L$.

   Without the condition (b), consider $g(x) \equiv 0$, $f(y) = \begin{cases} 1, & y = 0 \\ 0, & y \neq 0 \end{cases}$ and $x_0 = 0$. Then

$$y_0 = \lim_{x \to 0} g(x) = 0, \quad L = \lim_{y \to 0} f(y) = 0.$$

   However, $f(g(x)) \equiv 1$ and thus $\lim_{x \to 0} f(g(x)) = 1 \neq L$.

   Think about which step fails to be true in our proof if condition (b) is dropped.

   Remark: For more examples, please refer to **Question 5 in Part III** and **Ex 4.2.13** in the textbook. Notice that **there is a typo in the text**:

   $\lim_{x \to 1} g(f(x))$ and $g\left(\lim_{x \to 1} f(x)\right)$ should be changed to $\lim_{x \to 0} g(f(x))$ and $g\left(\lim_{x \to 0} f(x)\right)$ respectively.

Part II: Some comments.

In this chapter we begin the study of functions of one real variable. It is a necessary preparation for Chapter 5 and these two chapters are definitely the most important ones in this course. You need to be familiar with the material of this chapter, especially Section 4.1 and 4.2.

1. **Cluster point (limit point, accumulation point)**

   - A cluster point of a set of A of real numbers is a real number c that can be "approximated" by points of $\mathbb{R}$ in the sense that every neighborhood of c also contains a point of $\mathbb{R}$ other than c itself. Notice that **a cluster of a set A does itself not have to be an element of** A, as will be seen from the example below.
   - **Theorem 4.1.2** A number $c \in \mathbb{R}$ is a cluster point of A if and only if there exists a sequence (a_n) in A such that $\lim a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

   Example: Consider the set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$, then each element of A is an isolated point and 0 is a cluster point of A (but not contained in A). Please refer to **Example 4.1.3** to see more examples.

2. **Definition of the limit**

   You should realize the connection between Section 3.1 and 4.1. Here $\delta(\varepsilon)$ plays the same role as $K(\varepsilon)$ did in Section 3.1. Please notice that when discussing the limit of a function $f(x)$ at a cluster point $c \in \mathbb{R}$, **we do not care the value of $f(x)$ at c and we only consider the behavior of $f(x)$ as x approaches c**. In fact, it is even allowed that $f(x)$ is not defined at c.

   Please refer to **Example 4.1.7** and learn how to prove that $\lim\limits_{x \to c} f(x) = L$ by definition for given $f(x), c, L$. Pay special attention to the way that we determine $\delta(\varepsilon)$.

3. **Sequential criterion (Theorem 4.1.8)** is especially important. It is the bridge between limit of a sequence and limit of a function. The $\varepsilon - \delta$ definition of the limit is used to establish a limit, while sequences are more often used to evaluate a limit or prove that a limit fails to exist.

4. **Divergence Criteria**

   To prove that **a certain number** L is not the limit of a function $f(x)$ at a cluster point c, we can find a sequence (x_n) converging to c while $(f(x_n))$ does not converge to L.

   To prove that the function $f(x)$ **does not have a limit** at c, we can either find a sequence (x_n) converging to c such that $(f(x_n))$ is unbounded or find two sequences $(x_n), (y_n)$, both converging to c while $(f(x_n)), (f(y_n))$ converge to different real numbers (these are two different cases of a divergent sequence).

5. **Limit Theorems (boundedness, multiplication by a constant, addition, subtraction, product, quotient, order-preservation, squeeze, square root)** in Section 4.2 are similar to those in Section 3.2. While the proofs should be read carefully, our main interest is the application of these theorems to calculation of limits. **You should complete the proof to Theorem 4.2.4 yourselves by definition**.

   More results about Limit Theorems are given in Part III.

6. **(Generalization of Theorem 4.2.6) Order-preservation property**. Let $A \subset \mathbb{R}$, c be a cluster point of A and $f, g : A \to \mathbb{R}$. If f, g satisfy

   (a) $f(x) \leq g(x), \quad \forall x \in A, x \neq c$;

(b) $\lim_{x\to c} f(x) = a, \quad \lim_{x\to c} g(x) = b.$

Then $a \le b$.

Another edition of Order-preservation property (generalization of Theorem 4.2.9). Let $f : A \to \mathbb{R}$ and c is a cluster point of A. Suppose

$$a < \lim_{x\to c} f(x) = L < b,$$

then there exists $\delta > 0$ such that

$$a < f(x) < b, \quad \forall x \in A \cap (c-\delta, c+\delta), x \neq c.$$

Part III: Additional exercises.

1. **(Ex 4.1.14)** Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to c}(f(x))^2 = L$.

   (a) Show that if $L = 0$, then $\lim_{x\to c} f(x) = 0$.

   (b) Show by example that if $L \neq 0$, then f may not have a limit at c.

   Remark: This problem together with the next question is analogous to Question 1 in Part I of the tutorial notes on Feb 7.

   Proof:

   (a) From the assumption, $\forall \varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - c| < \delta$ we have $\left|(f(x))^2 - 0\right| < \varepsilon^2$, which is equivalent to $|f(x) - 0| < \varepsilon$.
   Therefore, $\lim_{x\to c} f(x) = 0$.

   (b) Consider $f(x) = \operatorname{sgn}(x)$ and then $\lim_{x\to 0}(f(x))^2 = 1$ while $\lim_{x\to 0} f(x)$ does not exist.

2. **(Ex 4.2.15)** Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster of A. In addition, suppose that $f(x) \ge 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $\left(\sqrt{f}\right)(x) := \sqrt{f(x)}$. If $\lim_{x\to c} f(x) = L$ exists, prove that $\lim_{x\to c} \sqrt{f} = \sqrt{L}$.

   Proof: This is analogous to **Theorem 3.2.10** and it can be proved by **Theorem 3.2.10** the **Sequential Criterion 4.1.8**. The details are omitted.

3. **(Absolute value rule)(Ex 4.2.14)** Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster of A. If $\lim_{x\to c} f(x) = L$ exists and if $|f|$ denotes the function for $x \in A$ by $|f|(x) := |f(x)|$, prove that $\lim_{x\to c} |f(x)| = |L|$.

   Proof: This problem is analogous to **Theorem 3.2.9** and the proof is omitted.

4. **(Maximum/Minimum value rule)** Let $A \subset \mathbb{R}$, let $f_1, f_2 : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster of A. If $\lim_{x\to c} f_1(x) = L_1$, $\lim_{x\to c} f_2(x) = L_2$ exist and if $\max(f_1, f_2)$ denotes the function for $x \in A$ by $\max(f_1, f_2)(x) := \max(f_1(x), f_2(x))$, prove that $\lim_{x\to c} \max(f_1, f_2) = \max(L_1, L_2)$.

   Proof: The conclusion is easily proved by noticing that

$$\max(f_1, f_2)(x) = \frac{f_1(x) + f_2(x) + |f_1(x) - f_2(x)|}{2}$$

   and using the **Absolute value rule**.

5. For the following $g, f : \mathbb{R} \to \mathbb{R}$ and $x_0, y_0 \in \mathbb{R}$, discuss the existence of $\lim_{x\to x_0} g(x)$, $\lim_{y\to y_0} f(y)$ and $\lim_{x\to x_0} f(g(x))$ respectively.
   - $g(x)$ is the Riemann function, $f(y)$ is the Dirichlet function and $x_0 = 0, y_0 = 0$.
   - $g(x) = x^2$, $f(y) = \begin{cases} y & \text{if } y \le 1 \\ y+1 & \text{if } y > 1 \end{cases}$ and $x_0 = 1, y_0 = 1$.
   - $g(x) = \operatorname{sgn}(x)$, $f(y) = y(1-y^2)$ and $x_0 = 0, y_0 = 0$.
   - $g(x)$ is the Dirichlet function, $f(y) = y$ and $x_0 = 0, y_0 = 1$.
   - Both of $g(x), f(y)$ are the Riemann function and $x_0 = 0, y_0 = 1$.
   - $g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, $f(y) = \operatorname{sgn}(y)$ and $x_0 = 0, y_0 = 0$.

6. **(Class Exercise 5)**. Use $\varepsilon - \delta$ definition to show that
$$\lim_{x\to x_0} \frac{f}{g}(x) = \frac{L}{M},$$
provided $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M \neq 0$. Here f and g are defined on $(a, x_0) \cup (x_0, b)$.

Proof. First from our assumptions, for $\varepsilon_1 = \dfrac{|M|}{2} > 0$, there exists $\delta_1 > 0$ such that whenever $x \in (a, b)$ and $0 < |x - x_0| < \delta_1$ we have
$$|g(x) - M| < \frac{|M|}{2} \Longrightarrow 0 < \frac{|M|}{2} < |g(x)| < \frac{3|M|}{2} \quad \text{(why?)}$$
$$\Longrightarrow 0 < \frac{2}{3|M|} < \frac{1}{|g(x)|} < \frac{2}{|M|}. \tag{1}$$
Now for any $\varepsilon > 0$, there exists $\delta_2 > 0$ such that whenever $x \in (a, b)$ and $0 < |x - x_0| < \delta_2$ we have
$$|f(x) - L| < \frac{\varepsilon|M|}{4}. \tag{2}$$
And there exists $\delta_3 > 0$ such that whenever $x \in (a, b)$ and $0 < |x - x_0| < \delta_3$ we have
$$|g(x) - M| < \frac{\varepsilon|M|^2}{4(|L|+1)}. \tag{3}$$
Take $\delta = \min(\delta_1, \delta_2, \delta_3)$ and if $x \in (a, b)$ and $0 < |x - x_0| < \delta_2$ then all (1),(2),(3) hold and we have
$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &= \frac{|Mf(x) - Lg(x)|}{|Mg(x)|} = \frac{|M(f(x) - L) - L(g(x) - M)|}{|Mg(x)|} \\
&\le \frac{|M(f(x) - L)| + |L(g(x) - M)|}{|Mg(x)|} = \frac{|f(x) - L|}{|g(x)|} + \frac{|g(x) - M|}{|g(x)|} \cdot \frac{|L|}{|M|} \\
&< \frac{\varepsilon|M|}{4} \cdot \frac{2}{|M|} + \frac{\varepsilon|M|^2}{4(|L|+1)} \cdot \frac{2}{|M|} \cdot \frac{|L|}{|M|} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$
Therefore, $\lim_{x\to x_0} \dfrac{f}{g}(x) = \dfrac{L}{M}$.